Lec 4: 09/09/2013

FRW Universe: Some Physical Effects of Expansion

We now discuss some physical effects due to expansion that have important observational consequences:

(1) Redshift of physical momentum. In a FRW Universe, an object is not subject to a net non-gravitational force because of homogeneity and isotropicity. Therefore all objects moving in a FRW background can be considered as freely falling objects. From the equivalence principle, we know that freely falling objects move along a geodesic. A "geodesic" is the path between two points that has the shortes distance. The distance $s \equiv \int ds$ follows from the metric:

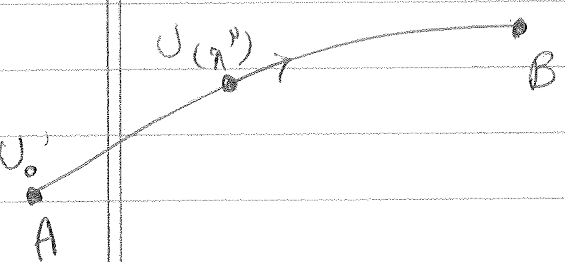
$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

For example, in Euclidean space (lik R^3) the geodesics are straight lines. As expected, freely moving particles in R^3 move along straight lines.

In general, the equation for a geodesic is given by:

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\kappa}\frac{dx^\nu}{du}\frac{dx^\kappa}{du} = 0$$

Here $\Gamma^\mu_{\nu\kappa}$ is the affine connection and u is a parameter that determines the position of a point on the geodesic;



For a massive particle, one can use the proper time τ as a physically good choice for u. Note that $ds^2 = c^2 d\tau^2$. In an FRW universe, as mentioned before, the affine connection takes a simple form. The equation for geodesic now becomes:

$$\frac{d^2x^i}{d\tau^2} = -\frac{2}{a(t)}\frac{da(t)}{dt}\frac{dx^i}{d\tau}\frac{dt}{d\tau}$$

x^i: spatial coordinates
t: time coordinate

This results in:

$$\frac{d}{dt}\left(\frac{dx^i}{d\tau}\right) = \frac{-2}{a}\frac{da}{dt}\frac{dx^i}{d\tau} \Rightarrow \frac{dx^i}{d\tau} \propto \frac{1}{a^2_{(t)}} \quad (I)$$

The physical momentum (i.e., $\vec{P}$) is given by:

$$P^i \propto m\, a_{(t)} \frac{dx^i}{d\tau}$$

Here $a_{(t)}$ is included to take the physical distance in an expanding universe into account. Equation (I) then leads to

$$\vec{P} \propto \frac{1}{a_{(t)}}$$

Although we have obtained this result for a massive particle, we can take the limit $m \to 0$ in a smooth manner. Therefore, momentum redshift $\propto a^{-1}_{(t)}$ is valid for both massive and massless particles.

In the case of a massless particle, $E = Pc$ ($P \equiv |\vec{P}|$). Therefore momentum redshift implies $E \propto a^{-1}_{(t)}$, while the speed of the particle remains constant ($= c$) at all times.

For a massive particle, we have:

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow v = \frac{p}{\sqrt{m^2 + \frac{p^2}{c^2}}}$$

If $p = p_0$, $v = v_0$ at the time when $a = a_0$, we will have:

$$v = \frac{p_0 \frac{a_0}{a}}{\sqrt{m^2 + \frac{p_0^2}{c^2}\left(\frac{a_0}{a}\right)^2}}$$

In the relativistic regime $p \gg mc$, we have:

$$v \simeq c$$

In this case, expansion mainly affects p but not v similar to that for a massless particle. However, momentum redshift implies that there will be a transition to the non-relativistic regime at some time when $\frac{a}{a_0} \sim \frac{p_0}{mc}$. From then on $p \lesssim mc$, which implies that:

$$v_{(t)} \propto \frac{1}{a_{(t)}}$$

This shows that a relativistically moving massive particle will eventually become non-relativistic in an expanding universe. Therefore transition from radiation domination (RD) to matter domination (MD) occurs dynamically. Initially, all elementary particles (massive or massless) are in the relativistic regime because of very high temperatures in the early universe. Later on, due to expansion, the massive particles (like the electron) become non-relativistic, while massless particles (like the photon) remain relativistic. The energy density in the non relativistic particles scale $\propto a^{-3}_{(t)}$, while that in the massless particles goes $\propto a^{-4}_{(t)}$. In consequence, as mentioned before, non-relativistic particles dominate at some point, which marks transition from RD to MD.

Another consequence of the momentum redshift in an expanding universe has to do with the temperature of a gas. If we have a gas of relativistic particles, the

$E = pc$. At equilibrium $\langle E \rangle = 3T$ for such a gas, which results in:

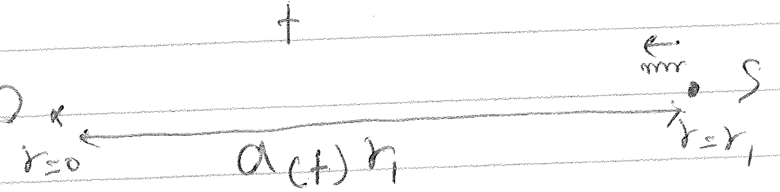
$$p \propto \frac{1}{a(t)} \Rightarrow T \propto \langle p \rangle \propto \frac{1}{a(t)}$$

On the other hand, in a gas consisting of non-relativistic particles in thermal equilibrium, temperature is related to the average momentum as follows:

$$\langle E_{kin} \rangle = 3T \Rightarrow \frac{\langle p^2 \rangle}{2m} = 3T \Rightarrow T \propto \langle p^2 \rangle \propto \frac{1}{a^2(t)}$$

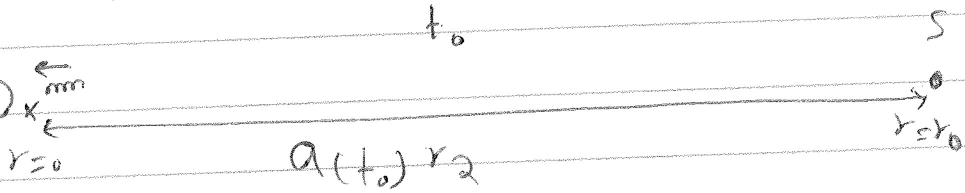
It is seen that the temperature redshifts faster for a non-relativistic gas.

(2) Frequency redshift. Expansion results in a decrease of any signal propagating in the universe. To demonstrate this, consider an observer at comoving coordinates $(0, \theta, \phi)$ and a source at comoving coordinates (r_1, θ, ϕ). The source emits a signal at time t moving at the speed of light



Source-observer distance at time $t = a_{(t)} r_1$

The signal is received at time t_0 by the observer:



Source-observer distance at time $t_0 = a_{(t_0)} r_1$

We note that since the source and observer have the same θ, ϕ coordinates, the distance between them does not depend on the space geometry (flat, closed, or open).

A particle moving at the speed of light has a trajectory that is a "null geodesic". A null geodesic is defined as a path along which $ds^2 = 0$. In a FRW universe, this implies:

$$-dt^2 + a^2(t)\left[dr^2 + \begin{Bmatrix} r^2 & k=0 \\ \sin^2 r & k=+1 \\ \sinh^2 r & k=-1 \end{Bmatrix} (d\theta^2 + \sin^2\theta \, d\phi^2)\right] = 0$$

Considering the source and observer in above, we have:

$$-dt^2 + a(t)^2\, dr^2 = 0 \Rightarrow dr = \frac{dt}{a(t)} \Rightarrow \int_{r_1}^{0} dr = -\int_{t}^{t_0} \frac{dt'}{a(t')} \Rightarrow$$

$$r_1 = \int_{t_0}^{t} \frac{dt'}{a(t')} \qquad (\mathrm{II})$$

Now consider a second signal that is sent by the source at the time $t+\delta t$ (where $\delta t \ll t$). This implies a frequency $f = \frac{1}{\delta t}$ as measured by the source. This signal is received by the observer at the time $t_0 + \delta t_0$, hence a frequency $f_0 = \frac{1}{\delta t_0}$ as measured by the observer. Since the comoving coordinates of the source and observer are $r=0$ and $r=r_1$ respectively regardless of the times at which the signals are sent and received, we have:

$$r_1 = \int_{t_0+\delta t}^{t+\delta t} \frac{dt'}{a(t')} \qquad (\mathrm{III})$$

Equations (II, III) result in:

$$\int_{t_0}^{t} \frac{dt'}{a(t')} = \int_{t_0+\delta t_0}^{t+\delta t} \frac{dt'}{a(t')} \Rightarrow \frac{\delta t}{a(t)} = \frac{\delta t_0}{a(t_0)} \Rightarrow \frac{\delta t}{\delta t_0} = \frac{a(t)}{a(t_0)}$$

$$\Rightarrow \frac{f_0}{f} = \frac{a(t)}{a(t_0)}$$

The redshift of the source z is defined as:

$$z = \frac{f}{f_0} - 1 \Rightarrow \frac{a(t_0)}{a(t)} = 1 + z$$

Since in an expanding universe $a(t_0) \geq a(t)$, then we have $f_0 \leq f$ implying that $z \geq 0$. The farther an object is, the longer it takes for the light to travel from the object to the observer. This implies that $a(t)$ was smaller at the time light left the farther objects, which are also older. Therefore, objects that are farther away are observed at a higher redshift z. For example, the CMB photons are a snapshot of the universe at a redshift $z \sim O(1000)$. Therefore, since $T \propto \frac{1}{a}$ for photons, the temperature of the CMB was $T \sim 3{,}000$ K at the time, called "epoch of recombination", which we will discuss in detail later.